Electrostatic Potential around a Charged Line

- Let

$$\rho(x, y, z) = q \delta(x) \delta(y)$$  \hspace{1cm} (1)

be the charge distribution, where $$\delta(x)$$ is the Dirac's delta function,

$$\int_{-\infty}^{\infty} \delta(x) \forall f(x) = f(0)$$  \hspace{1cm} (2)

and $$q$$ has the dimension of charge/length.

- The electrostatic potential $$\Phi(x, y, z)$$ (it won't depend on $$z$$) is determined from

$$\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \Phi(x, y, z) = -4\pi \rho(x, y, z)$$  \hspace{1cm} (3)

Let the gradient vector be

$$\nabla = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right)$$  \hspace{1cm} (4)

and Laplacian

$$\nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$  \hspace{1cm} (5)

Then, Eq. (3) can be rewritten as

$$\nabla^2 \Phi(x, y) = -4\pi q \delta(x) \delta(y)$$  \hspace{1cm} (6)
Let's integrate Eq.(6) in the circular area with radius \( R \).

\[
\iint_V \nabla^2 \phi(x, y) = -4\pi \int_V \theta(x) \theta(y) = -4\pi \tag{7}
\]

Now use Gauss' theorem. \( \int_V \nabla \cdot \mathbf{V} = \iint_S \mathbf{V} \cdot \mathbf{n} \)

\[
\iint_V \nabla \cdot \nabla \phi(x, y) = \iint_S \nabla \phi(x, y) \cdot \mathbf{n} \tag{8}
\]

where \( ds \) is the surface areal element vector normal to the surface. Note if the surfaces are normal to the \( x \) axis,

\[
\iint_V \nabla \cdot \mathbf{V} = \left( \int_a^b \frac{\partial}{\partial x} df \right) \mathbf{n}, \text{ only look at } x\text{-component of the vector}
\]

\[
\left( \int_a^b \frac{\partial}{\partial x} df \right) = \left( \int_a^b \frac{\partial}{\partial x} df \right) = \left( b f(b) - a f(a) \right) = \left( b f(b) + (-a) f(a) \right) = \iint_S d\mathbf{s} f
\]

Substituting Eq.(8) in (7)

\[
\iint_S \nabla \phi(x, y) = -4\pi \tag{9}
\]

From the symmetry, the gradient vector is normal to the surface with strength \( \frac{d\phi}{dR} \) uniform across the circle

\[
\therefore \ 2\pi R \frac{d\phi}{dR} = -4\pi \tag{10}
\]

\[
\therefore \ \frac{d\phi}{dR} = -\frac{2\pi}{R} \tag{11}
\]
\[ \phi(R) = -2\pi \log R + C \]  \hspace{1cm} (12)

where \( C \) is the integration constant. Setting 
\( C = 2\pi \log R_0 \),
\[ \phi(R) = -2\pi \log \left( \frac{R}{R_0} \right) \]  \hspace{1cm} (13)

![Graph](image)

- Measure the length in unit of \( R_0 \) and \(+2\pi \to 0\) (how we measure the charge density in 2D)

\[ \phi(x,y) = \frac{1}{2} q \log (r^2+y^2) \]  \hspace{1cm} (14)

Let \( z = x+i\,y = r e^{i\theta} \) where \( r = \sqrt{x^2+y^2} \) \( \tan \theta = y/x \) then

\[ \phi(x,y) = \frac{1}{2} \text{Re} \, q \log z \]  \hspace{1cm} (15)

\[ \therefore \log (re^{i\theta}) = \log r + i\theta. \]

This is the use of complex \( \log z \) to compute 2D electrostatic potential.